Characteristic Independence of Betti Numbers of Monomial Ideals in Five Variables

With a brief outlook on seven and eight variables

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Fixed monomials; a changing coefficient field

Fix monomials $g_1, \ldots, g_r$, and set

$$S_k = k[x_1, \ldots, x_n], \qquad M_k = (g_1, \ldots, g_r).$$

A minimal multigraded free resolution has the form

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S_k/M_k \longrightarrow 0.$$

The multigraded Betti number $\beta_{i,m}(k)$ counts the minimal free generators in homological degree i and multidegree m.

The question

Can these numbers change when only $\mathrm{char}(k)$ changes?

Throughout, Betti numbers are those of the quotient S_k/M_k.

Five variables are characteristic-independent

Theorem (Ripke–Y., 2026)

If $S = k[x_1, \dots, x_5]$ and $M \subseteq S$ is a monomial ideal, then the multigraded, graded, and total Betti numbers of S/M are independent of $\mathrm{char}(k)$.

Variables	Dependence?	Source
$n \leq 4$	Cannot occur	Alesandroni
$n = 5$	Cannot occur	Ripke–Y.
$n = 6$	Can occur	projective-plane example

Six is the smallest number of variables in which dependence can occur.

The proof strategy

Step 1: Prove the squarefree case

Assume first that M is squarefree.

Stanley–Reisner
squarefree ideals correspond to simplicial complexes

$\rightarrow$

Hochster's formula
turns Betti numbers into homology dimensions

$\rightarrow$

Five-vertex topology
gives torsion-free integral homology

$\boxed{\text{torsion-free} \Longrightarrow \text{characteristic-independent}}$

Step 2: Return to arbitrary monomial ideals

For each relevant multidegree, use the squarefree twin reduction to reduce the problem to the squarefree case without increasing support.

The squarefree bridge: Hochster's formula

For a simplicial complex Δ on $[n]$,

$$I_\Delta = (x_F : F \notin \Delta), \qquad x_F = \prod_{j \in F} x_j.$$

Minimal nonfaces correspond to minimal squarefree generators.

Hochster's formula (see [MS05], Cor. 5.12)

For $W \subseteq [n]$,

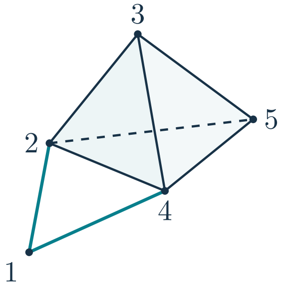
$$\boxed{\beta_{i,x_W}(S/I_\Delta) = \dim_k \widetilde{H}_{|W|-i-1}(\Delta_W; k)}$$

where

$$\Delta_W = \{F \in \Delta : F \subseteq W\}.$$

A Betti number question becomes a homology question.

A five-vertex example



Tetrahedron boundary; triangle 124 is not filled.

$$I_\Delta = (x_1x_3,\ x_1x_5,\ x_1x_2x_4,\ x_2x_3x_4x_5).$$

$$\Delta \simeq S^2 \vee S^1.$$

At full support $m = x_1x_2x_3x_4x_5$,

$$\beta_{2,m} = \dim_k \widetilde{H}_2(\Delta; k) = 1,$$

$$\beta_{3,m} = \dim_k \widetilde{H}_1(\Delta; k) = 1.$$

These values are the same over every field.

Why torsion is the obstruction

The Universal Coefficient Theorem (see [Hat02], Theorem 3A.3) gives

$$0 \to \widetilde{H}_q(\Delta;\mathbb{Z}) \otimes_{\mathbb{Z}} k \to \widetilde{H}_q(\Delta;k) \to \operatorname{Tor}_1^{\mathbb{Z}}(\widetilde{H}_{q-1}(\Delta;\mathbb{Z}), k) \to 0.$$

Integral homology	Over a field k
$\mathbb{Z}^r$	contributes k^r in every characteristic
p-torsion	can change homology in characteristic p

Consequence

If all reduced integral homology groups are torsion-free, then

$$\widetilde{H}_q(\Delta;\mathbb{Z}) \cong \mathbb{Z}^{r_q} \quad \Longrightarrow \quad \widetilde{H}_q(\Delta;k) \cong k^{r_q}$$

for every field k. Thus the homology dimensions are independent of the characteristic.

Topology on at most five vertices

Govc–Marzantowicz–Michalak–Pavešić (2025, Proposition 3.4)

Every connected simplicial complex on at most five vertices is homotopy equivalent to a wedge of spheres.

Wedges of spheres have free abelian integral homology, so there is no torsion. For example,

$$\widetilde{H}_2(S^2 \vee S^1; \mathbb{Z}) \cong \mathbb{Z}, \qquad \widetilde{H}_1(S^2 \vee S^1; \mathbb{Z}) \cong \mathbb{Z}.$$

If an induced subcomplex has c connected components, then

$$\widetilde{H}_0 \cong \mathbb{Z}^{c-1},$$

while for $q \geq 1$ its homology is the direct sum of the homology of its connected components.

Thus every induced subcomplex on at most five vertices has torsion-free integral homology.

A squarefree replacement, one multidegree at a time

Ordinary polarization makes an ideal squarefree, but it can introduce many new variables. Instead, fix a Taylor multidegree m.

1. **Restrict to generators dividing** m (Gasharov–Hibi–Peeva, 2002)
   Keep only the minimal generators that divide m.
2. **Keep only exponent maxima** (Alesandroni, 2023)
   For each surviving generator, remember only which coordinates attain the exponents appearing in m.
3. Replace those attained coordinates by squarefree variables, producing the squarefree twin M''_m.

$$\boxed{M \xrightarrow[\text{GHP}]{\text{restrict at } m} M_m \xrightarrow[\text{Alesandroni}]{\text{twin}} M'_m \xrightarrow{\text{squarefree}} M''_m}$$

Keep only the information relevant to the chosen multidegree m.

The squarefree twin preserves the Betti number

Let

$$T_m = k[y_j : j \in \operatorname{supp}(m)], \qquad y_m = \prod_{j \in \operatorname{supp}(m)} y_j.$$

Squarefree twin reduction (Alesandroni; see Ripke–Y., 2026, Theorem 2.2)

For every Taylor multidegree m,

$$\boxed{\beta^S_{i,m}(S/M) = \beta^{T_m}_{i,y_m}(T_m/M''_m)}.$$

The squarefree twin M''_m uses exactly the variables indexed by $\operatorname{supp}(m)$, so it never uses more variables than the original ideal.

We preserve the relevant Betti number without increasing support size.

An illustrative squarefree twin calculation

Take

$$M = (x_1^3 x_2^2 x_3,\ x_1^3 x_2 x_3^2,\ x_1^2 x_2^2 x_3^2,\ x_4^4) \subseteq k[x_1, \ldots, x_5],$$
$$m = x_1^3 x_2^2 x_3^2, \qquad W = \{1, 2, 3\}.$$

Generator dividing m	Squarefree twin
$x_1^3 x_2^2 x_3$	$y_1 y_2$
$x_1^3 x_2 x_3^2$	$y_1 y_3$
$x_1^2 x_2^2 x_3^2$	$y_2 y_3$

x_4^4 is discarded before forming the twin.

$$T_m = k[y_1, y_2, y_3],$$
$$M''_m = (y_1 y_2, y_1 y_3, y_2 y_3).$$

Its complex consists of three isolated vertices.

$$\beta_{2,m}(S/M) = \dim_k \widetilde{H}_0(\Delta; k) = 2.$$

Finishing the proof

For every relevant multidegree m,

$$\beta^S_{i,m}(S/M) = \beta^{T_m}_{i,y_m}(T_m/M''_m).$$

Since

$$|\operatorname{supp}(m)| \leq 5,$$

the right-hand side comes from a squarefree ideal in at most five variables.

$$\boxed{\text{squarefree five-variable case} \Longrightarrow \beta_{i,m}(S/M) \text{ is characteristic-independent.}}$$

Finally,

$$\beta_{i,j} = \sum_{\deg m = j} \beta_{i,m}, \qquad \beta_i = \sum_j \beta_{i,j}.$$

Therefore the multigraded, graded, and total Betti numbers are all characteristic-independent.

Six vertices: the first torsion obstruction

Reisner's example (1976): the six-vertex triangulation of $\mathbb{RP}^2$ has

$$\widetilde{H}_1(\Delta;\mathbb{Z}) \cong \mathbb{Z}/2, \qquad \widetilde{H}_2(\Delta;\mathbb{Z}) = 0.$$

In characteristic 2, UCT gives

$$\dim_k \widetilde{H}_1(\Delta;k) = 1, \qquad \dim_k \widetilde{H}_2(\Delta;k) = 1.$$

Outside characteristic 2, both dimensions are zero.

At full support $|W| = 6$, Hochster gives

$$\boxed{\beta_{4,6} = 1, \qquad \beta_{3,6} = 1 \quad \text{only in characteristic } 2.}$$

Six-variable consequence

For every monomial ideal in six variables, all characteristics other than 2 give the same Betti numbers. (The five-variable bound is sharp.)

A finite catalogue of squarefree ideals in five variables

Squarefree monomial ideals correspond to antichains in $2^{[5]}$. ([RY26], Proposition 8.1)

7581 labeled antichains → **210** relabeling classes → **208** nonzero proper ideals → **134** graded Betti tables

The two removed classes are (0) and (1).

Macaulay2 and Python/Hochster computations agree.

The topology proves independence; the computation records the table types.

What happens next?

A prime-by-prime threshold (Ongoing work)

For a prime p, let $\tau(p)$ be the smallest number of variables in which characteristic-p dependence can occur.

By the squarefree twin reduction, Hochster's formula, and UCT, this is equivalently

$$\boxed{\tau(p) = \min\{|V(K)| : \widetilde{H}_q(K;\mathbb{Z}) \text{ has } p\text{-torsion for some } q\}.}$$

We already know $\tau(2) = 6$.

For an odd prime there are two tasks:

$$\begin{aligned} \text{eight-vertex case exists} &\implies \tau(p) \leq 8, \\ \text{seven-vertex obstruction} &\implies \tau(p) \geq 8. \end{aligned}$$

Can odd-prime dependence first appear at eight variables?

Characteristic three: the eight-vertex upper bound (Ongoing work)

Karoubi–Weibel input ([GMMP25], Corollary 4.10)

$$\boxed{\mathrm{KW}(\mathbb{Z}/3) = 8.}$$

For our purposes, this gives an eight-vertex simplicial complex K with

$$\pi_1(K) \cong \mathbb{Z}/3.$$

First homology is the abelianization of the fundamental group:

$$H_1(K;\mathbb{Z}) \cong \pi_1(K)_{\mathrm{ab}} \cong \mathbb{Z}/3.$$

Thus eight vertices support 3-torsion, and hence

$$\boxed{\tau(3) \leq 8.}$$

Seven vertices: controlling first homology (Ongoing work)

For a connected simplicial complex K on at most seven vertices, GMMP's classification gives ([GMMP25], Theorem 3.10)

$$\pi_1(K) \cong F_r, \qquad \mathbb{Z}/2 * F_r, \qquad \text{or} \qquad \mathbb{Z} \times \mathbb{Z}.$$

Since

$$H_1(K;\mathbb{Z}) \cong \pi_1(K)_{\mathrm{ab}},$$

the possibilities are

$$\mathbb{Z}^r, \qquad \mathbb{Z}/2 \oplus \mathbb{Z}^r, \qquad \mathbb{Z}^2.$$

Therefore

$$\boxed{H_1(K;\mathbb{Z}) \text{ has no odd torsion.}}$$

If K is disconnected, apply this componentwise; $H_0(K;\mathbb{Z})$ is always free abelian.

Seven vertices: higher homology via $K^\vee$ (Ongoing work)

For a simplicial complex K on $[n]$, define its Alexander dual by

$$K^\vee = \{F \subseteq [n] : [n] \setminus F \notin K\}.$$

For $q \geq 2$, combinatorial Alexander duality ([BT09], Theorem 1.1) gives

$$\widetilde{H}_q(K;\mathbb{Z}) \cong \widetilde{H}^{\,n-q-3}(K^\vee;\mathbb{Z}).$$

Since $n \leq 7$, $n - q - 3 \leq 2$.

Thus, in the extremal case $n = 7$,

Homology of K	Cohomology of $K^\vee$
$\widetilde{H}_2(K)$	$\widetilde{H}^2(K^\vee)$
$\widetilde{H}_3(K)$	$\widetilde{H}^1(K^\vee)$
$\widetilde{H}_4(K)$	$\widetilde{H}^0(K^\vee)$
$q \geq 5$	degree ≤ -1

Seven vertices: low-degree cohomology (Ongoing work)

Cohomological UCT (see [Hat02], Theorem 3.2) shows:

- H^0 and H^1 are torsion-free.
- Any torsion in $H^2(K^\vee;\mathbb{Z})$ comes from torsion in $H_1(K^\vee;\mathbb{Z})$.

In negative reduced degrees, $\widetilde{H}^{-1}$ is 0 or $\mathbb{Z}$, and lower degrees vanish, so no torsion occurs there.

But $K^\vee$ also has at most seven vertices.

So the same GMMP classification applies to $K^\vee$ and rules out odd torsion in $H_1(K^\vee;\mathbb{Z})$.

Therefore no higher homology group of K can contain odd torsion.

$$\boxed{\text{No simplicial complex on at most seven vertices has odd torsion.}}$$

$\tau(p) \geq 8$ for every odd prime p.

Eight vertices reach the odd-prime bound (Ongoing work)

Prime	Eight-vertex input	Integral torsion
3	$\mathrm{KW}(\mathbb{Z}/3) = 8$	$H_1(K;\mathbb{Z}) \cong \mathbb{Z}/3$
5	explicit complex K_5	$H_2(K_5;\mathbb{Z}) \cong \mathbb{Z}/5$
7	explicit complex K_7	$H_2(K_7;\mathbb{Z}) \cong \mathbb{Z}/7$

For $p = 5, 7$, the existing cases have complete two-skeletons and 35 selected tetrahedra; exact integral boundary calculations certify the displayed H_2-torsion.

Together with the seven-vertex lower bound,

$$\boxed{\tau(3) = \tau(5) = \tau(7) = 8.}$$

Conclusion

Published five-variable theorem

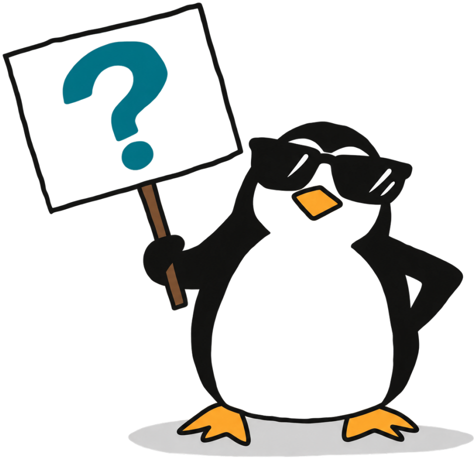
For every monomial ideal in at most five variables, the multigraded, graded, and total Betti numbers are independent of the characteristic.
The six-variable projective-plane example makes the bound sharp.

Ongoing work: prime-by-prime thresholds

$$\boxed{\tau(2) = 6, \qquad \tau(3) = \tau(5) = \tau(7) = 8.}$$

Characteristic dependence becomes a minimum-vertex torsion problem.

Questions?



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Addendum: the 5- and 7-torsion certificate

For $p = 5, 7$, let K_p be an eight-vertex complex with

$$f(K_p) = (8, 28, 56, 35),$$

consisting of the complete two-skeleton together with 35 selected tetrahedra.

Two-cycles

The complete two-skeleton has

$$\ker \partial_2 \cong \mathbb{Z}^{35}.$$

The 35 tetrahedral boundaries therefore give

$$A_p : \mathbb{Z}^{35} \longrightarrow \mathbb{Z}^{35}.$$

Integral certificate

For the explicit choices,

$$|\det A_5| = 5, \qquad |\det A_7| = 7.$$

Hence

$$H_2(K_p; \mathbb{Z}) = \operatorname{coker} A_p,$$

and therefore

$$\boxed{H_2(K_5; \mathbb{Z}) \cong \mathbb{Z}/5, \qquad H_2(K_7; \mathbb{Z}) \cong \mathbb{Z}/7.}$$

Since $\det A_p \neq 0$, $H_3(K_p; \mathbb{Z}) = 0$.

Addendum: complete eight-variable Betti table

For $p = 5$ or 7, the nonzero graded Betti numbers of S/I_{K_p} are:

(i, j)	$\operatorname{char} k \neq p$	$\operatorname{char} k = p$
$(0, 0)$	1	1
$(1, 4)$	35	35
$(2, 5)$	84	84
$(3, 6)$	70	70
$(4, 7)$	20	20
$(4, 8)$	0	1
$(5, 8)$	0	1