Lattice Models for Quantum Superalgebras

Colors and Supercolors

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Overview

1. Introduction and Motivations
2. Strategy
3. Results and Conjecture

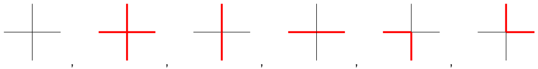
Lattice Model

Definition (Informal)

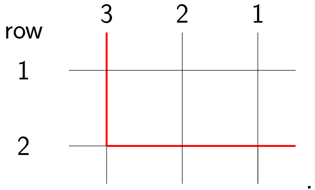
A lattice model $\mathcal{L}$ is an $n \times m$ grid with its edges filled according to vertex table.

Example:

- For example, we might use this vertex table (6 vertices):



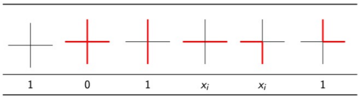
  for which fillings become paths going from the top and exiting to the right
- Models are indexed by external (boundary) edges.
- For example, here's one possible filling for the indicated boundary:

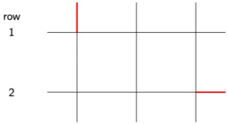


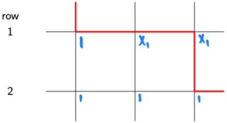
Definition (Partition Function)

Given a lattice model with fixed boundary conditions, the partition function $\mathcal{Z}(\mathcal{L})$ of the lattice is the weighted sum of *all* admissible states, which are paths with non-zero weight.

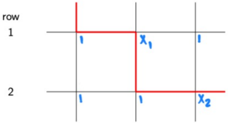
Example: Weights for our six vertices. Parameters x_i depend on row i in which the vertex appears.



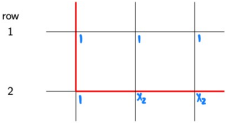




x_1^2



$x_1 x_2$



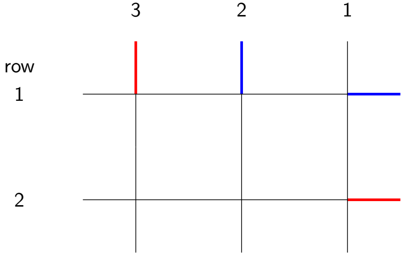
x_2^2

In this case: $\mathcal{Z}(\mathcal{L}) = x_1^2 + x_1 x_2 + x_2^2$, a polynomial in two variables x_1, x_2.

Lattice Models with Colored Paths

Some models allow for multiple colors - one for each of m rows. We describe the top boundary by a partition and the right-hand boundary by a permutation $w \in S_m$ (indicating the order of colors from top to bottom).

Example. Take $m = 2$ with $\lambda = (3, 2)$ and $w = (12)$



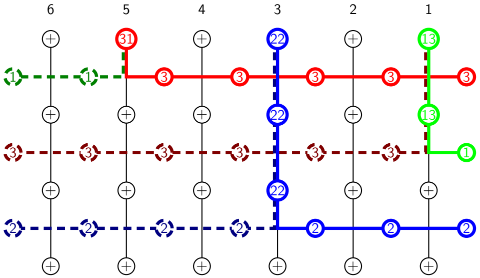
Here the coloring order is Red $>$ Blue, and $w = (12)$ indicates we've reversed the ordering, top to bottom, on the right-hand boundary.

Super-Lattice Model

Definition (Informal)

A *super-lattice model* has colored edges (moving downward, rightward) and super-colored edges (moving downward, leftward). On vertical edges, colors and supercolors are paired. Thus, model boundaries are indexed by one partition λ and two permutations $w, w' \in S_m$.

Example. Below is an admissible state with $\lambda = (5, 3, 1)$, $w = (312)$, and $w' = (132)$. Note that w is the right boundary condition and w' is the left boundary condition.

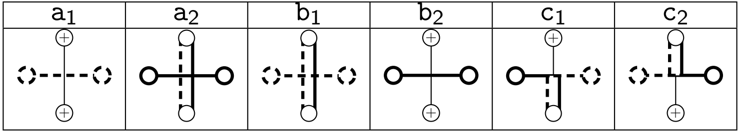


Vertices for the Super-Lattice Model

The rules:

- Colors move down, right; Supercolors move down, left
- At most one color or supercolor on a horizontal edge
- Colors and supercolors travel in pairs on vertical edges

give us the following possible vertices (for any color, supercolor):



Note: We give the weights [3] generic names for later reference.

Research Questions

Motivating Questions

- For given λ, w, w', what does the set of all admissible fillings count?
- Are there interesting bijections with families of combinatorial objects?
- Can we find interesting weights and study the partition functions of these models $\mathcal{L}_{\lambda,w,w'}$? Use them to prove conjectures about (in)famous families of polynomials...

Past work:

- Previous Polymath Jr. projects have "solved" models with one partition λ and one permutation w. (e.g., [2], to appear in *IMRN*)

For experts: The rules for the given model were chosen based on quantum superalgebra modules (and their so-called R-matrices)

- Further Question: How does the choice of superalgebra module (and its corresponding weights) affect the resulting family of partition functions?
- Ultimately connected to supersymmetries between bosons and fermions

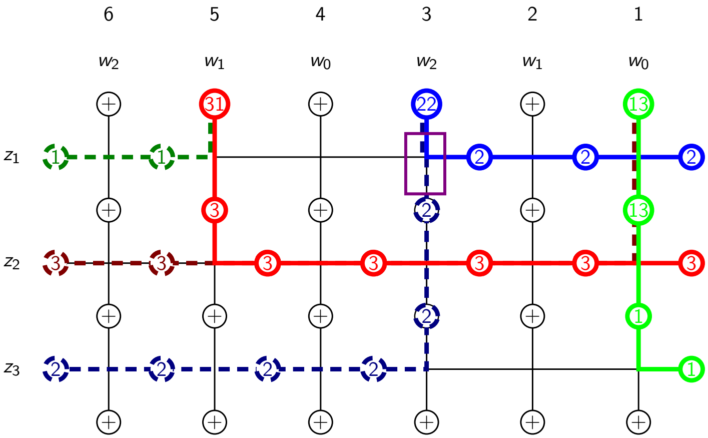
Research Strategy

- **Goal.** Compute the partition function of $\mathcal{L}_{\lambda,w,w'}$ for all $\lambda \vdash m$ and $w, w' \in S_n$.
- **First steps.**
  1. Identify w, w' with $\mathcal{L}_{\lambda,w,w'} = 0$.
  2. Identify w, w' such that $\mathcal{L}_{\lambda,w,w'}$ has a unique admissible state.
  3. Compute remaining partition functions recursively by relating permutation index pairs (train argument).
- **Dream.** Understand the quantum group module for super-lattice models

Question (rephrased).

- What is the minimal set of states we need to compute to know all partition functions?
- And what do their partition functions look like?

An Inadmissible (Vanishing) State



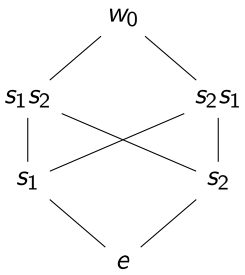
Strong order and Coxeter combinatorics

Recall the following definitions from [1]

- the **length** of w is the number of inversion: $\ell(w) = \#\{\, (i < j) \mid w(i) > w(j)\}$.
- $u \leq v$ **in strong order** if $u \cdot t_{i_1 j_1} t_{i_2 j_2} \ldots t_{i_k j_k} = v$, where $t_{i_1 j_1} = (i_1\ j_1)$.

Example:

length	elements
0	$e = 123$
1	$213 = s_1, \quad 132 = s_2$
2	$231 = s_1 s_2, \quad 312 = s_2 s_1$
3	$321 = s_1 s_2 s_1 = s_2 s_1 s_2 = w_0.$



Note that s_1 and s_2 are incomparable, so are $s_1 s_2$ and $s_2 s_1$.

Vanishing Conjecture

Let $|\mathcal{L}_{\lambda,w,w_0 u}|$ be the number of admissible states.

Conjecture

Let $\lambda \vdash m$, $w, u \in S_n$, and w_0 the longest word in S_n. Then

$$|\mathcal{L}_{\lambda,w,w_0 u}| = \begin{cases} 0, & u < w \text{ in strong order} \\ 1, & u = w \end{cases}.$$

Example:

$$\begin{array}{c c} w \backslash w_0 u & \begin{array}{cccccc} w_0 & s_2 s_1 & s_1 s_2 & s_1 & s_2 & e \end{array} \\ \begin{array}{c} e \\ s_2 \\ s_1 \\ s_1 s_2 \\ s_2 s_1 \\ w_0 \end{array} & \begin{pmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & * & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{*} & \color{red}{1} & \color{red}{*} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Vanishing / one-state table

Observations

- Diagonal of ones.
- Zeros lie strictly to the left of that diagonal.
- Almost–lower triangular.
- The circled entry should have an admissible state, but the conjecture does not capture it.

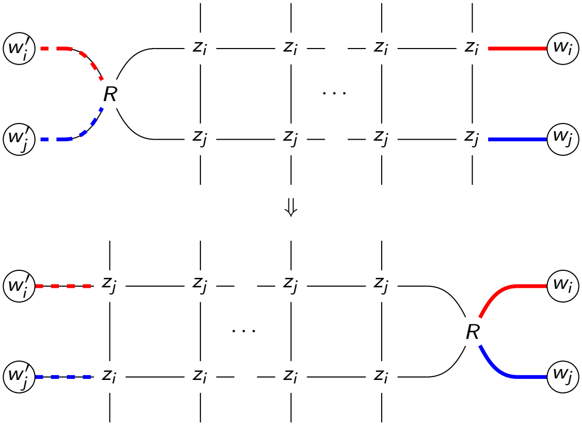
$$
\begin{array}{c|cccccc}
 & w_0 & s_2 s_1 & s_1 s_2 & s_1 & s_2 & e \\
\hline
e & 1 & * & * & * & * & * \\
s_2 & 0 & 1 & * & * & * & * \\
s_1 & 0 & * & 1 & * & * & * \\
s_1 s_2 & 0 & 0 & 0 & 1 & * & * \\
s_2 s_1 & 0 & 0 & 0 & * & 1 & * \\
w_0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

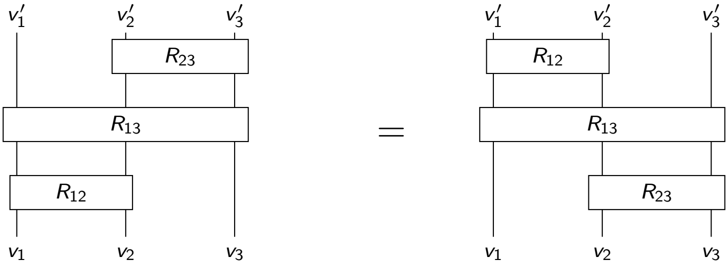
1	exactly one state ($u = w$),
0	vanishes ($u < w$ in strong order),
$*$	conjecture does not constrain this pair.

Train argument for color/scolor model

Notation:

- w_i represents the **color** decorated at row i under permutation of w, and w_i' represents the **scolor** decorated at row i under permutation of w'.
- z_i and z_j marks the row number before and after the run-through of R-vertex.

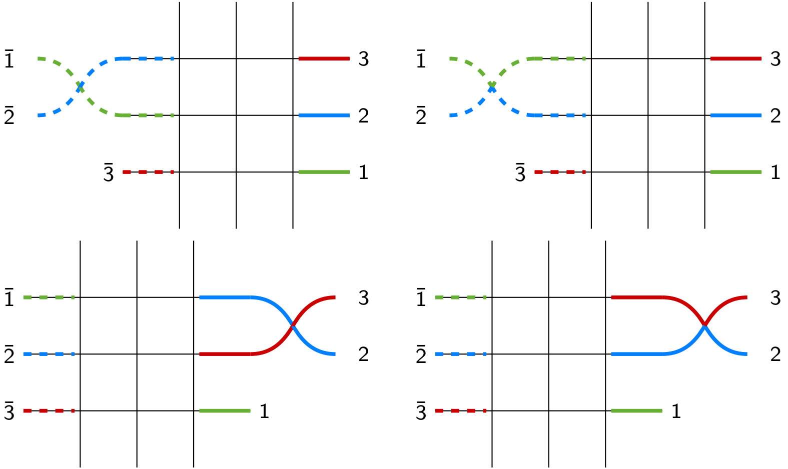




The Yang-Baxter equation: $R_{23}\, R_{13}\, R_{12} \;=\; R_{12}\, R_{13}\, R_{23}$ i.e. the two orderings yield the same linear map on $V_1 \otimes V_2 \otimes V_3$, with R**'s generated from the quantum group**

Example of train argument on 3×3 Super-lattice model

For simplification, it seems like for a 3×3 lattice model, but one can image the arbitrary rows above and below with arbitrary column in the middle. The lattice models would be, from left to right, $\mathcal{L}_{\lambda,w_0,s_1}, \mathcal{L}_{\lambda,w_0,e}, \mathcal{L}_{\lambda,s_1 s_2,e}, \mathcal{L}_{\lambda,w_0,e}$.



The relations from train argument on the super-lattice model

Case $(w = w_0, w' = e)$

$$q\,(z_1^3 - z_2^3)\,\boxed{Z\big(\mathcal{L}_{\lambda,w_0,s_1}\big)} + (1-q^2)\,z_2^2\,z_1\,\boxed{Z\big(\mathcal{L}_{\lambda,w_0,e}\big)}$$
$$= s_1\Big[(z_1^3 - z_2^3)\,Z\big(\mathcal{L}_{\lambda,s_1 s_2,e}\big) + (1-q^2)\,z_1^3\,\boxed{Z\big(\mathcal{L}_{\lambda,w_0,e}\big)}\Big],$$

$$Z\big(\mathcal{L}_{\lambda,w_0,s_1}\big) = 0, \quad Z\big(\mathcal{L}_{\lambda,w_0,e}\big) = z_1^{\lambda_1-2}\,z_2^{\lambda_2-1}\,z_3^{\lambda_3}.$$

Upshot

- Use the vanishing result $Z(\mathcal{L}_{\lambda,\ldots}) = 0$ to eliminate terms.
- Substitute the already known partition function formula.
- Only $Z(\mathcal{L}_{\lambda,s_1 s_2,e})$ remains as an unknown.
- Solve for $Z(\mathcal{L}_{\lambda,s_1 s_2,e})$ (an entry marked unknown in the table).

Note: s_1 simply means to flip z_1 and z_2. For example, if $f = z_1^2 + z_2$, then $s_1 f = z_2^2 + z_1$.

In the context of the table

$$
\begin{array}{c}
\begin{array}{cccccc} w_0 & s_2 s_1 & s_1 s_2 & s_1 & s_2 & e \end{array} \\
\begin{array}{c} e \\ s_2 \\ s_1 \\ s_1 s_2 \\ s_2 s_1 \\ w_0 \end{array}
\begin{pmatrix}
1 & * & * & * & * & * \\
0 & 1 & * & * & * & * \\
0 & * & 1 & * & * & * \\
0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & * & 1 & * \\
0 & 0 & 0 & \boxed{0} & 0 & \boxed{1}
\end{pmatrix}
\end{array}
$$

$Z(\mathcal{L}_{\lambda, s_1 s_2, e})$ corresponds to the circled entry, which is one of the partition functions that we do not know yet (we only know the partition functions corresponding with entries of either 0 or 1). We utilized the boxed entries to find the circled entry.

Research Questions

Motivating Questions

- For given λ, w, w', what do the set of all admissible fillings count?
- Are there interesting bijections with families of combinatorial objects?

Alternating Sign Matrices

Definition.

An **alternating sign matrix** (ASM) is an $n \times n$ matrix with entries of $-1, 0, 1$ such that each column and row sum to 1 with non-zero alternating sign entries.

Alternating sign matrices are a generalization of permutation matrices. In particular, permutation matrices are the ASMs with no -1s.

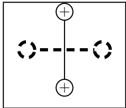
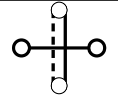
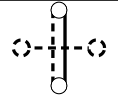
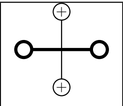
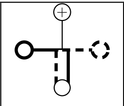
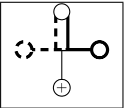
Examples.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

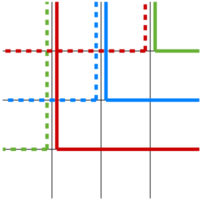
Non-Example.

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

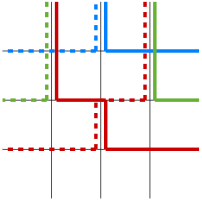
Weight assigned to each possible super-lattice vertex:

					
0	0	0	0	-1	1

Example.



$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



$$\Leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

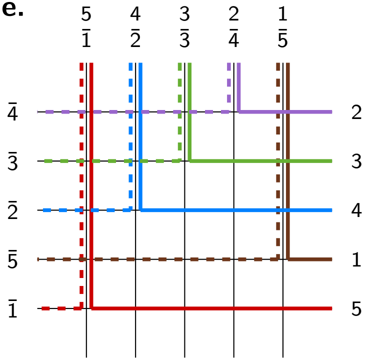
Trivial Boundary Condition Case

Proposition

For $w \in S_n$, $\mathcal{L}_{\lambda,w,w_0w}$ have a unique non-zero admissible state.

- These lattice models are in bijection with the permutation matrices. Numbers without bars indicate solid colors and numbers with bars indicate dashed colors.

Example.



$$\Leftrightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

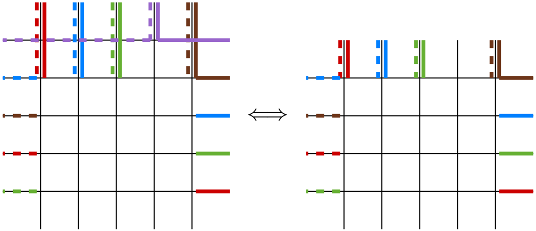
$(n-1) \times n$ and $n \times n$ lattices

Idea.

The color/scolor exiting at the first row of a square lattice model need to be coming from the same column.

Proposition.

Given top boundary λ, the number of admissible states of $n \times n$ lattice model $\mathcal{S}_\lambda$ equals to the number of admissible states of $(n-1) \times n$ lattice model $\mathcal{L}_{\lambda'}$ such that λ' has one pair of color and scolor decoration less then λ.



Criteria for non-zero admissible states

We use the previous ideas to make a final conjecture:

$$\begin{cases} \text{Bijective alternating sign matrix} \implies \text{Lattice States} \\ \text{Trivial boundary condition case} \implies \text{Foundation for train argument} \\ (n-1) \times n \text{ and } n \times n \text{ lattice} \implies \text{Enabled analysis on reduced lattice} \end{cases}$$

With these tools for proof in mind, we revisit the vanishing condition proposed earlier and formulate a more refined criterion to further investigate the cases that were undetermined.

Criteria for non-zero admissible states

Idea: Which pair of λ, w', w satisfy $\mathcal{Z}(\mathcal{L}_{\lambda,w',w}) \neq 0$?

Let s be a permutation which $s = \prod_{i=1}^{k} \sigma_i$ and $S_i := \{x \in \mathbb{Z}_n : \sigma_i(x) \neq x\}$, where $\sigma_i \in S_n$ are all disjoint permutations ordered in the sequence: $\forall i < j$, $\exists x \in S_i$ such that x appeared eailier than all $y \in S_j$ in the lattice left condition from top to bottom row.

Conjecture

When permutations w, s *(as above)* satisfy $l(sw) = l(s) + l(w)$, then $\mathcal{L}_{(\lambda, w_0 w, sw)}$ has a non-zero admissible state if and only if

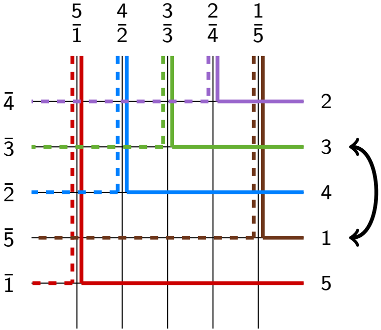
(i) $w >_B sw$ in strong order; (*rephrased vanishing condition*)

(ii) Consider set $Z_i := \{\min(S_i), \min(S_i)+1, ..., \max(S_j)-1, \max(S_j)\}$ and accumulated exiting set $E_i = \{x \in S_j : \forall j < i\}$. Then, $\forall i$, $|(Z_i/S_i) \cap E_i| \geq l(s_i)$. (*exiting condition*)

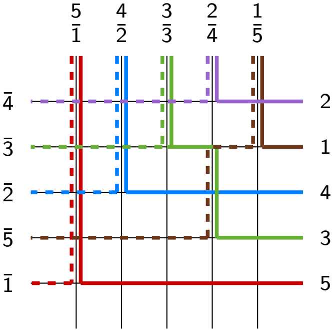
Note: $l(s)$ is the Coxeter length of the cycle.

Demonstration of the conjecture

	Simple permutation pair	**Extra transposition**
λ	$(5,4,3,2,1)$	$(5,4,3,2,1)$
w, s	$(1432), e$	$(1432), (13) \implies sw = (12)(34)$



$\implies$



More specifically, (1432) is a cover of $(12)(34)$. Note $S = \{1,3\}$, $Z_i = \{2,4\}$, and $Z_i' = \{1,2,3\}$.

Recap

Questions.

- What is the partition function of the super-lattice model?
- Are there any known combinatorial objects in bijection to these lattice models?

Results.

- Special cases with monostate
  - Alternating Sign Matrices
- Conjecture for non-vanishing states

Next Steps.

- Determine all boundary conditions with unique and multiple states in the square lattice model.
- Find the operator between two arbitrary partition functions for super-lattice models.

References

[1] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Vol. 231. Graduate Texts in Mathematics. Springer, 2005. DOI: 10.1007/3-540-27596-7.

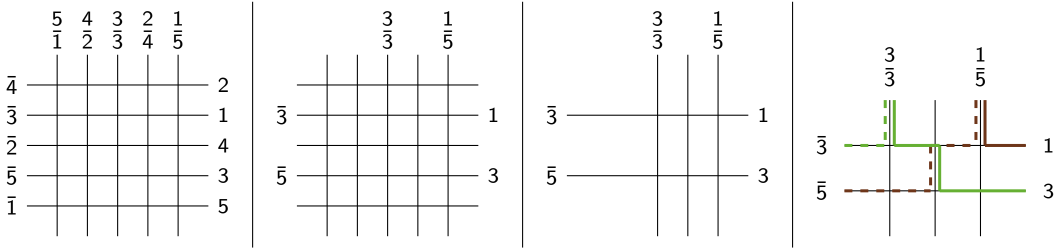
[2] Ben Brubaker et al. *Kirillov's conjecture on Hecke-Grothendieck polynomials*. 2024. arXiv: 2410.07960 [math.CO]. URL: https://arxiv.org/abs/2410.07960.

[3] Ben Brubaker et al. *Metaplectic Iwahori Whittaker functions and supersymmetric lattice models*. 2024. arXiv: 2012.15778. URL: https://arxiv.org/abs/2012.15778.

Thank you!

Sketch of Proof

The idea of proof is through decomposition of lattice based on the disjoint cycles.



For the sake of simplification we want to reduce the lattice simultaneously as we consider disjoint cycle. While the row removal is trivial, column is a bit complicated:

- check if col_{i+1} and col_{i-1} has non empty top condition.
- If both of the adjacent columns has non empty top condition, check the next column without top condition.
- If either of the adjacent columns has empty top condition and both color and scolor corresponded , remove col_i.

Sketch of Proof

To prove our conjecture, we need to prove criteria $\Longleftrightarrow$ admissible states.

Suppose there is a admissible states given left and nontrivial right boundary condition (permutation from w on the right), then the idea is to show that all $\mathcal{L}_1, ..., \mathcal{L}_k$ need to be admissible, which we for now assume to indicate the fulfillment of the two criteria.

Suppose otherwise, then we assume it granted the admissible of all $\mathcal{L}_1, ..., \mathcal{L}_k \implies$ the admissible of the initial lattice.

Therefore, it suffices to prove the criteria for every disjointed cycle rather than s.