What Does Small Growth Tell Us About Algebras?

Phillip Yoon
Hunter College
Hunter Mathematics and Statistics Colloquium
Joint Work with Be'eri Greenfeld

From groups to algebras

- The *Cayley graph* of a finitely generated group records multiplication by generators.
- The radius-n ball is

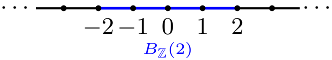
$$B_G(n) = \{g \in G : \ |g| \le n\}.$$

- For $\mathbb{Z}$, $\quad |B_{\mathbb{Z}}(n)| = 2n + 1 \quad$ (linear growth).
- For $\mathbb{Z}^d$, $\quad |B_{\mathbb{Z}^d}(n)| \asymp n^d$.

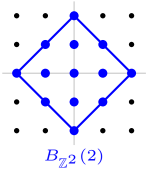
Gromov's theorem

Polynomial growth $\iff$ virtually nilpotent.

Virtually nilpotent groups are nilpotent up to finite index, so they form a restricted, well-understood class.



Cayley graph of $\mathbb{Z}$.



Cayley graph of $\mathbb{Z}^d$
(shown for $d = 2$).

Growth in algebras: examples first

Let A be an associative algebra over some field k and $V \subseteq A$ a finite-dimensional generating subspace with $1 \in V$.

Growth function

$$\gamma_A(n) = \dim_k(V^n),$$

where V^n is the span of products of at most n generators.

- $k[x]$: $\quad \dim_k(V^n) = n + 1$ (linear growth)
- $k[x, y]$: $\quad \dim_k(V^n) = \binom{n+2}{2}$ (quadratic growth)
- $k[x_1, \ldots, x_d]$: $\quad \dim_k(V^n) = \binom{n+d}{d} \asymp n^d$

In the commutative finitely generated case, this growth degree agrees with *Krull dimension*. In the noncommutative case, the picture is subtler.

What is known about polynomial growth?

Degree of polynomial growth	**Growth regime**	**What is known**
0	bounded	finite-dimensional
1	linear	restricted structure; Small–Stafford–Warfield imply PI behavior
$(1,2)$	between linear and quadratic	impossible (Bergman's gap theorem)
≥ 2	quadratic and above	no further gaps; all polynomial growth degrees occur (Borho–Kraft)

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≤ 6	known counterexample to L'vov	Bell–Small–Smoktunowicz (2012): nil counterexample over a countable field with growth roughly n^6
2	this work	quadratic growth with no largest nilpotent ideal

Question (L'vov, Dniester Notebook, 4th ed., 1993)

If A is finitely generated and has polynomial growth, must A have a *largest nilpotent ideal*?

Counterexample to L'vov's Question

Theorem (Greenfeld–Yoon, 2026)

There exists a finitely generated monomial algebra A_W such that

$$\gamma_{A_W}(n) = n^2\left(\frac{1}{2} + o(1)\right),$$

A_W has *no largest nilpotent ideal*, and both the quadratic order n^2 and the coefficient $\frac{1}{2}$ are best possible.

So the counterexample works:

- over an arbitrary field, and
- at the slowest possible quadratic threshold.

Idea

Build a monomial algebra from one carefully chosen infinite word and read both growth and ideal structure from its subwords.

A monomial algebra from a word

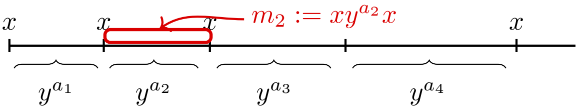
Start with an infinite word

$$W = xy^{a_1}xy^{a_2}xy^{a_3}\cdots \qquad (a_1 < a_2 < a_3 < \cdots \to \infty).$$

- Begin with the free algebra $k\langle x, y\rangle$.
- Kill every word that is *not* a contiguous subword of W.
- The quotient is a monomial algebra A_W.
- The surviving monomials are exactly the contiguous subwords of W.

Key point

Growth becomes a counting problem: how many subwords of length at most n occur in W?



Why there is no largest nilpotent ideal

- Each block $m_i := xy^{a_i}x$ appears only once in W, so the ideal it generates is nilpotent.
- Hence the algebra contains infinitely many nilpotent ideals, one for each isolated block.
- Any *largest* nilpotent ideal N would have to contain all of them.
- But different blocks can be chained along the word to produce nonzero products of arbitrary length.

$$\begin{aligned} m_1 &= xy^{a_1}x && \in N \\ m_2 &= xy^{a_2}x && \in N \\ m_3 &= xy^{a_3}x && \in N \\ &\vdots \downarrow \end{aligned}$$

$$m_1 y^{a_2} m_3 y^{a_4} \cdots y^{a_{2t-2}} m_{2t-1}$$

is an initial segment of W, so it is nonzero and lies in N^t.

Conclusion

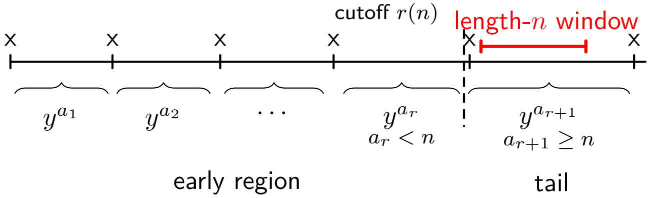
$N^t \neq 0$ for every t, so N cannot be nilpotent. Therefore no largest nilpotent ideal exists.

Why the growth is quadratic

For fixed n, choose $r = r(n)$ so that

$$a_r < n \le a_{r+1}.$$

Then split W into an *early region* and a *tail*.



- A length-n subword starts either in the early region or in the tail.
- In the tail, once the gaps exceed n, the possible windows are very limited.
- The number of length-n subwords is therefore linear in n.
- Summing over lengths $1, 2, \ldots, n$ gives quadratic growth with leading term $\frac{1}{2}n^2$.

Conclusion

- In groups, polynomial growth forces restricted structure.
- In commutative algebras, growth reflects ordinary algebraic dimension (Krull dimension).
- In noncommutative algebras, even quadratic growth can still allow unexpected ideal-theoretic behavior.

Thank you!