Finding Equations to the Parametrized Curves of Affine Variety

Phillip Yoon, Hunter College

Mentor: Valeriy Sergeev

CUNY GRADUATE CENTER

Abstract

An affine variety in $\mathbb{R}^n$ is a subset that is a vanishing set of some collection of polynomials. When is a parametrized curve in $\mathbb{R}^3$ an affine variety and how do we find the equations defining it, if it is indeed an affine variety? In this project, we study what affine varieties and ideals are in the context of algebraic geometry, look at the ideas behind the division algorithm in $\mathbb{R}[x_1, x_2, \cdots, x_n]$, and give an example on how to determine whether the parametrized curve is an affine variety or not and find the ideal for such a curve.

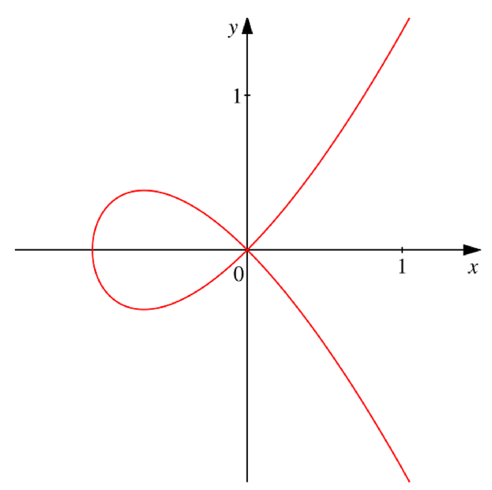

Figure 1. An example of an affine variety in $\mathbb{R}^2$.

Background

We introduce some definitions and ideas that are relevant for the scope of the project:

- **Affine Variety:** Let k be a field, and let $f_1, f_2, \cdots, f_s$ be polynomials in a polynomial ring $k[x_1, \cdots, x_n]$. Then we set $\mathbf{V}(f_1, \cdots, f_s) = \{(a_1, \cdots, a_n) \in k^n \mid f_i(a_1, \cdots, a_n) = 0 \text{ for all } 1 \le i \le s\}$. We call $\mathbf{V}(f_1, \cdots, f_s)$ the **affine variety** defined by $f_1, \cdots, f_s$.
  In other words, affine variety $\mathbf{V}(f_1, \cdots, f_s)$ is the set of all solutions of the system of equations $f(x_1, \cdots, x_n) = \cdots = f_s(x_1, \cdots, x_n) = 0$.
  For example, the above curve is the variety $\mathbf{V}(y^2 - x^2(x+1))$. A unit circle would be the variety $\mathbf{V}(x^2 + y^2 - 1)$.
- **Ideal:** A subset $I \subseteq k[x_1, \cdots, x_n]$ is an **ideal** if it satisfies:
  - $0 \in I$.
  - If $f, g \in I$, then $f + g \in I$.
  - If $f \in I$ and $h \in k[x_1, \cdots, x_n]$, then $hf \in I$.
- Let $f_1, \cdots, f_s$ be polynomials in $k[x_1, \cdots, x_n]$. Then we set
  $$< f_1, \cdots, f_s > = \left\{ \sum_{i=1}^{s} h_i f_i \mid h_1, \cdots, h_s \in k[x_1, \cdots, x_n] \right\}$$
  The crucial fact is that $< f_1, \cdots, f_s >$ is an ideal.
- If $f_1, \cdots, f_s \in k[x_1, \cdots, x_n]$, then $< f_1, \cdots, f_s >$ is an ideal of $k[x_1, \cdots, x_n]$. We call $< f_1, \cdots, f_s >$ the **ideal generated by** $f_1, \cdots, f_s$.

Parametrizations of Affine Variety become crucial when there's infinitely many solutions and we cannot write down all the solutions. For instance, the parameterization for the above curve is $x = t^2 - 1, y = t(t^2 - 1)$. This is a more effective way to give a solution since there's infinitely many solutions for the above curve. Now, given a parametrization, such as this one, **how do we know if this curve is an affine variety?** This question is at the crux of the project.

The Main Problem

Let $V \subseteq \mathbb{R}^3$ be the curve parametrized by $(t, t^m, t^n), n, m \ge 2$. How do we determine if this parametrized curve is indeed an affine variety? Can we also determine the polynomials that generate the ideal for the affine variety?

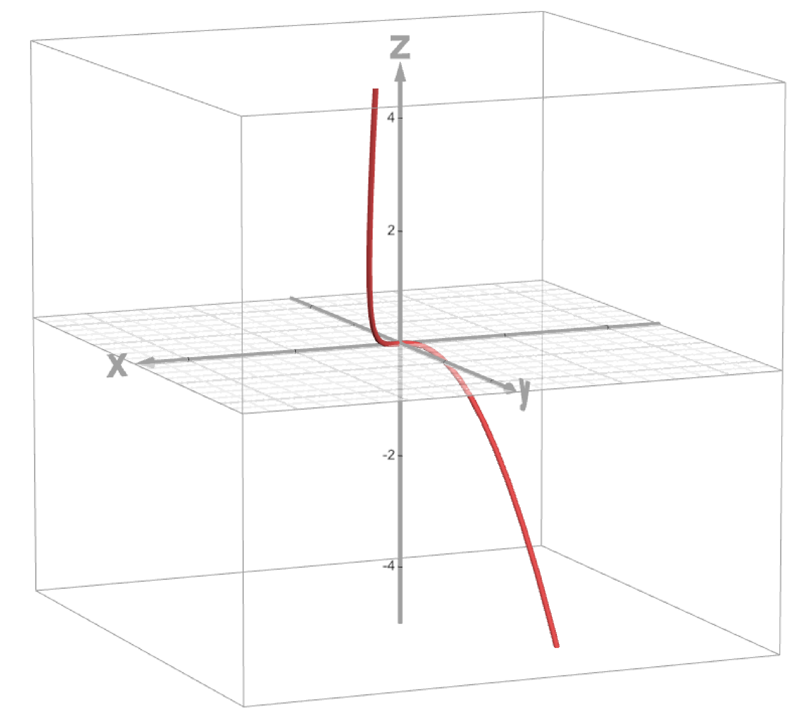

Figure 2. The visualization of the given curve parametrized by $(t, t^m, t^n), m = 2, n = 3$.

Terminology and Definition for the Solution

Here, we introduce some relevant terminology and definition that will be crucial in solving for the problem:

Lexicographic Order. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be in $\alpha \in \mathbb{Z}^n_{\ge 0}$. We say $\alpha >_{lex} \beta$ if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. We will write $x^\alpha >_{lex} x^\beta$ if $\alpha >_{lex} \beta$.

1. The **multidegree** of f is
   $$\text{multideg}(f) = \max(\alpha \in \mathbb{Z}^n_{\ge 0} \mid a_\alpha \neq 0)$$
   (The maximum is taken with respect to >).
2. The **leading coefficient** of f is
   $$\text{LC}(f) = a_{\text{multideg}(f)} \in k.$$
3. The **leading term** of f is
   $$\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f).$$

Example: let $f = 4xy^2z + 4z^2 - 5x^3 + 7x^2y^2$ and let > denote lex order. Then, multideg(f) = $(3, 0, 0)$, LC(f) = -5, LT(f) = $-5x^3$.

Solution

To determine if the given curve is an affine variety, we start with the claim that $V = \mathbf{V}(y - x^m, z - x^n)$. If $(t, t^m, t^n) \in V$, then evaluating the given polynomials gives: $y - x^m = t^m - (t)^m = 0$ and $z - x^n = t^n - (t)^n = 0$. This proves that $(t, t^m, t^n) \in \mathbf{V}(y - x^m, z - x^n)$, which means that $y - x^m$ and $z - x^n$ vanish on V.

In addition, any polynomials of the form $h_1(y - x^m)$ and $h_2(z - x^n)$ also vanish. Now, let $f = h_1(y - x^m) + h_2(z - x^n) + r$. Division Algorithm for $k[x_1, \cdots, x_n]$ says that monomials of r are not divisible by LT($y - x^m$) and LT($z - x^n$), which respectively have to be y and z for $r \in \mathbb{R}[x]$ (every term of the remainder is a monomial in just x). However, for this to happen, we cannot assume the usual lex order of $x > y > z$, but rather, we must consider the inverse order $z > y > x$.

Let $f(x, y, z) \in \mathbb{R}[x, y, z]$ vanishing on V. By the division algorithm, with lex given with $z > y > x$, we can write:

$$f(x, y, z) = h_1(y - x^m) + h_2(z - x^n) + r(x)$$

Since f vanished on V, for all $t \in \mathbb{R}$, we have $0 = f(t, t^m, t^n) = r(t)$. Given that $r(t)$ must also vanish on all real number, $r = 0$, which in turn means that $f \in \langle y - x^m, z - x^n \rangle$. This shows that $\underline{\mathbf{I}(V) = \langle y - x^m, z - x^n \rangle}$, the ideal of all functions vanishing on V.

Aside: Division Algorithm in $k[x_1, \cdots, x_n]$

Division Algorithm in $k[x_1, \cdots, x_n]$**.**

Let > be a monomial order on $\mathbb{Z}^n_{\ge 0}$ and let $F = (f_1, \cdots, f_s)$ be an ordered s-tuple of polynomials in $k[x_1, \cdots, x_n]$. Then every $f \in k[x_1, \cdots, x_n]$ can be written as $f = q_1 f_1 + \cdots + q_s f_s + r$ where $q_i, r \in k[x_1, \cdots, x_n]$, and either $r = 0$ or r is a linear combination, with coefficients in k, of monomials, none of which is divisible by any of LT(f_1) , $\cdots$, LT(f_s). We call r a **remainder** of f on division by F. Furthermore, if $q_i f_i \neq 0$, then multideg(f) $\ge$ multideg($q_i f_i$).

Example.

Let us divide $f = x^2y + xy^2 + y^2$ by $f_1 = y^2 - 1$ and $f_2 = xy - 1$ with lex order with $x > y$.

$$\begin{array}{rlll}
q_1: & x + 1 & & \\
q_2: & x & & \underline{\quad r \quad} \\
\begin{array}{r} y^2 - 1 \\ xy - 1 \end{array} & \overline{\big) \, x^2y + xy^2 + y^2} & & \\
& \underline{x^2y - x} & & \\
& \quad xy^2 + x + y^2 & & \\
& \quad \underline{xy^2 - x} & & \\
& \qquad \underline{2x + y^2} & & \\
& \qquad\quad y^2 & \to 2x & \\
& \qquad\quad \underline{y^2 - 1} & & \\
& \qquad\qquad \underline{1} & & \\
& \qquad\qquad 0 & \to 2x + 1 &
\end{array}$$

This shows that

$$x^2y + xy^2 + y^2 = (x + 1) \cdot (y^2 - 1) + x \cdot (xy - 1) + 2x + 1.$$

Other visualizations of the given curve

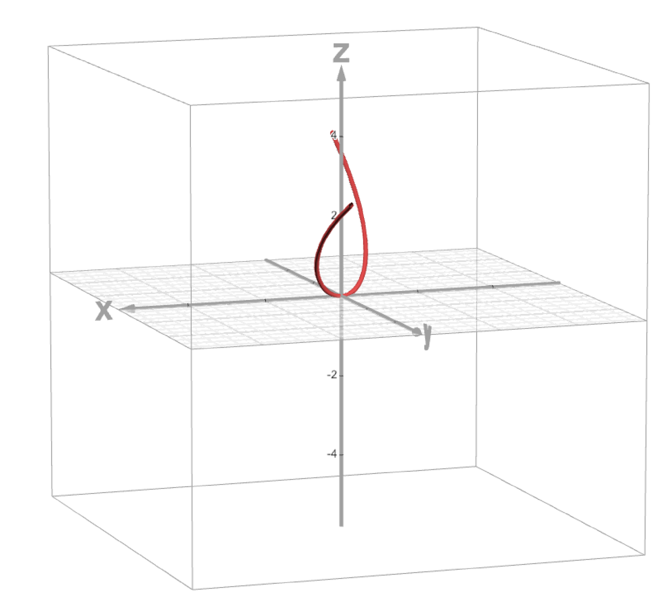

Figure 3. Visualization of the curve when $m = 3, n = 2$.

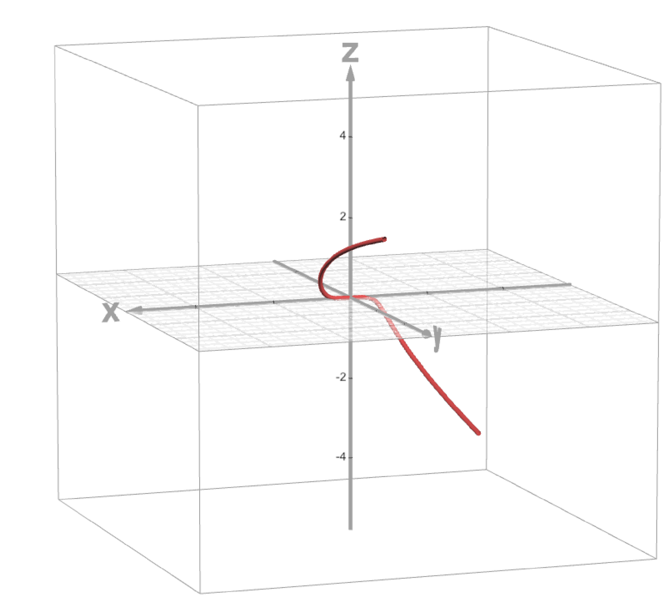

Figure 4. Visualization of the curve when $m = 10, n = 5$.

References

[1] Donal O'Shea David A. Cox and John B. Little.
Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra (undergraduate texts in mathematics).
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Directed Reading Program Presentation