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1. Introduction and preliminaries

The posed problem is as follows: Show that it is impossible to construct an equilateral triangle in the plane with vertices that are lattice points. An equilateral triangle has two properties: equal sides and equal angles. That is, all three sides of an equilateral triangle have the same length and each interior angle of an equilateral triangle is 60 degrees. A lattice point is a point in a Cartesian coordinate system where both the x- and y-coordinates are integers. For instance, $(2, 4)$ is a lattice point, but $(\frac{2}{3}, \sqrt{2})$ is not a lattice point.

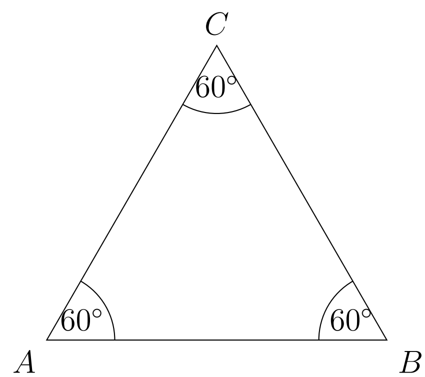

Figure 1. An equilateral triangle.

1.1. **Categorizing the types of edges.** On a Cartesian plane with lattice points, there are four types of edges that can be drawn: a vertical edge, a horizontal edge, an edge with a positive slope, and an edge with a negative slope. In other words, every line segment joining two lattice points has either slope 0, undefined, positive, or negative, so these four categories exhaust all possibilities. For the purpose of this paper, I will name them category 1, 2, 3, and 4 respectively:

(1) Category 1: vertical lines with vertices (a, b) and (a, c), where $a, b, c \in \mathbb{Z}$.
(2) Category 2: horizontal lines with vertices (a, b) and (c, b), where $a, b, c \in \mathbb{Z}$.
(3) Category 3: lines with vertices (a, b) and (c, d), where $a, b, c, d \in \mathbb{Z}$ and $m = \frac{d-b}{c-a} > 0$.
(4) Category 4: lines with vertices (a, b) and (c, d), where $a, b, c, d \in \mathbb{Z}$ and $m = \frac{d-b}{c-a} < 0$.

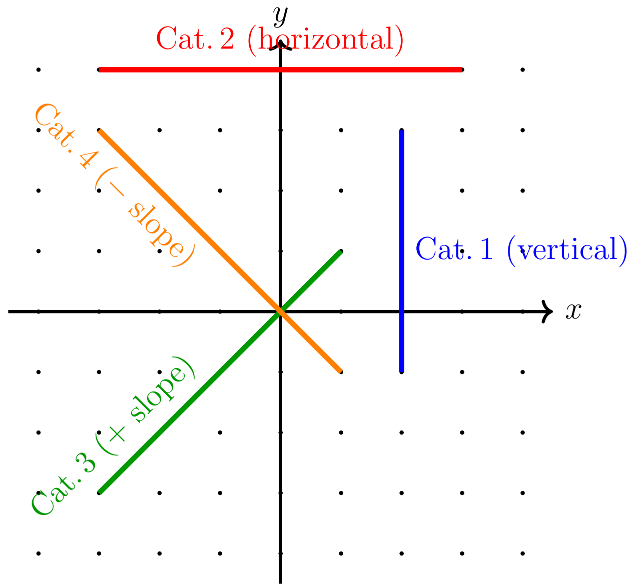

FIGURE 2. Examples of each category possible for lattice points on a Cartesian plane.

1.2. **Categorizing the types of triangles.** Now that I have established the possible categories of edges, I can combine these categories to make triangles on a Cartesian plane. As for the possibility of combinations of 4 categories, I need to choose 3 categories since triangles have 3 edges. Replacement is possible, since a repetition of category does not constitute that they are the same edge, but rather, they are the same 'kind' of edge. Order does not matter, as different order of categories can simply be a matter of which edges I would be drawing first. This is a multiset problem, and therefore the number of possibilities can be computed as follows:

$$\binom{n+k-1}{k} = \binom{4+3-1}{3} = \binom{6}{3} = 20.$$

Then, I can list all 20 possibilities of triangles based on the categories as follows:

1. $\{1,1,1\}$	11. $\{2,2,2\}$
2. $\{1,1,2\}$	12. $\{2,2,3\}$
3. $\{1,1,3\}$	13. $\{2,2,4\}$
4. $\{1,1,4\}$	14. $\{2,3,3\}$
5. $\{1,2,2\}$	15. $\{2,3,4\}$
6. $\{1,2,3\}$	16. $\{2,4,4\}$
7. $\{1,2,4\}$	17. $\{3,3,3\}$
8. $\{1,3,3\}$	18. $\{3,3,4\}$
9. $\{1,3,4\}$	19. $\{3,4,4\}$
10. $\{1,4,4\}$	20. $\{4,4,4\}$

Based on the selection, some possibilities can be ruled out right away. The multisets {1,1,1} and {2,2,2} are impossible choices for forming triangles, since they are all parallel to each other. Similarly, with any multisets with repeating entries of 1 or 2 are impossible choices as well, since two out of three edges will be parallel to each other. However, that does not hold for repeating entries of 3 or 4, since they can have different slope values despite being in the same category. Therefore, the following multisets are also impossible choices for forming triangles: {1,1,2}, {1,1,3}, {1,1,4}, {1,2,2}, {2,2,3}, and {2,2,4}.

Since there are 8 entries omitted from the original list of 20 entries, now the only possibilities I consider are as follows:

1. $\{1,2,3\}$	7. $\{2,3,4\}$
2. $\{1,2,4\}$	8. $\{2,4,4\}$
3. $\{1,3,3\}$	9. $\{3,3,3\}$
4. $\{1,3,4\}$	10. $\{3,3,4\}$
5. $\{1,4,4\}$	11. $\{3,4,4\}$
6. $\{2,3,3\}$	12. $\{4,4,4\}$

The strategy from here on is to exhaust all possibilities of triangles that are made from all possibilities of edges, which has been done in this section, and prove, case by case, that forming an equilateral triangle in the plane with vertices that are lattice points is impossible. Except for Case 1, every case will include some contradiction in regards to rational numbers and irrational numbers. Although there are 12 possible edge-length combinations, they collapse into fewer than 12 distinct cases due to symmetry, which will be demonstrated in the next section.

2. Proof, case by case

One can observe that many of the aforementioned triangle types are equivalent under symmetries of the square lattice. Specifically, the group of rigid motions that preserve the integer lattice $\mathbb{Z}^2$ is the dihedral group D_4, which consists of four rotations (by 0°, 90°, 180°, and 270°) and four reflections (across the x-axis, y-axis, and the lines $y = x$ and $y = -x$).

These transformations preserve distances and lattice point structure, so any triangle configuration that can be mapped to another by a D_4 action is effectively the same for the purposes of the proof. This can be shown by Lemma 2.1:

Lemma 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry satisfying $f\big(\mathbb{Z}^2\big) \subseteq \mathbb{Z}^2$. Then there exist a vector $v \in \mathbb{Z}^2$ and an integral orthogonal matrix $A \in \mathrm{Mat}_2(\mathbb{Z})$ such that*

$$f(x) = Ax + v \qquad (x \in \mathbb{R}^2),$$

and A is one of the eight signed-permutation matrices

$$I_2,\ \rho_{90},\ \rho_{180},\ \rho_{270},\ R_x,\ R_y,\ R_{y=x},\ R_{y=-x},$$

i.e. the elements of the dihedral group D_4 acting on the square lattice. In other words, translations, the four reflections, and the three quarter-turn rotations are the only rigid motions of the square lattice.[1]

[1]Rigid motions are geometric transformations that preserve the size and shape of a figure.

Proof. Orthogonality forces each entry of A to be 0 or ±1, and the dot-product conditions then leave precisely the eight signed permutation matrices listed above. Composing with any integer translation completes the description. $\square$

By Lemma 2.1, triangles that differ by a reflection (a mirror image) or 90° rotations are congruent in the lattice, so it is possible to group the 12 valid triangle types into D_4-*orbits* under this action, reducing the total number of distinct triangle cases that need to be analyzed.[2]

Then, the aforementioned 12 combinations can be reduced down to 5 cases:

$$\begin{aligned}
&\text{Case 1:}\ \ \mathrm{Orb}_{D_4}\big(\{1,2,3\}\big) = \big\{\{1,2,3\},\{1,2,4\}\big\},\\
&\text{Case 2:}\ \ \mathrm{Orb}_{D_4}\big(\{1,3,3\}\big) = \big\{\{1,3,3\},\{1,4,4\},\{2,3,3\},\{2,4,4\}\big\},\\
&\text{Case 3:}\ \ \mathrm{Orb}_{D_4}\big(\{1,3,4\}\big) = \big\{\{1,3,4\},\{2,3,4\}\big\},\\
&\text{Case 4:}\ \ \mathrm{Orb}_{D_4}\big(\{3,3,3\}\big) = \big\{\{3,3,3\},\{4,4,4\}\big\},\\
&\text{Case 5:}\ \ \mathrm{Orb}_{D_4}\big(\{3,3,4\}\big) = \big\{\{3,3,4\},\{3,4,4\}\big\}.
\end{aligned}$$

In each subsection, I will explain why these combinations can be seen as the same type of triangle, and prove, for each case, that forming an equilateral triangle is impossible. For the following cases, it will be assumed that three vertices of any given triangles are distinct and that the slopes of edges are distinct so that the triangle is non-degenerate. In addition, it should be noted that every slope between lattice points is a rational number, since the slope values will be in a form of $\frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0$.

2.1. **Case 1.** Since category 1 is a vertical line and category 2 is a horizontal line, any triangle with edges of category 1 and 2 inevitably have to be a right triangle, as can be seen in Figure 3. However, given that the definition of an equilateral triangle includes that all angles are equal to 60 degrees, it is not possible for triangles in case 1 to be equilateral triangles, given that one of the angles inevitably has to be 90 degrees. In addition, it can also be seen that the triangle formed with {1,2,3} is basically a mirror image of {1,2,4}, rendering them to be included in the same case, by Lemma 2.1.

2.2. **Case 2.** I will first visualize the first two triangles in this case to demonstrate why they are to be included in the same case.

[2] D_4 is the dihedral group of order 8, consisted of 4 rotations and 4 reflections that map a square onto itself.

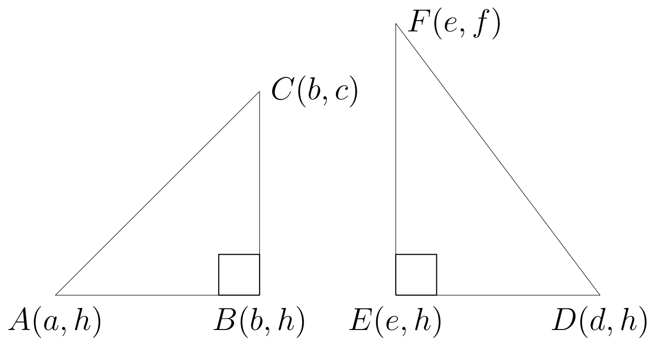

FIGURE 3. Visualization of triangles made by {1,2,3} (left) and {1,2,4} (right).

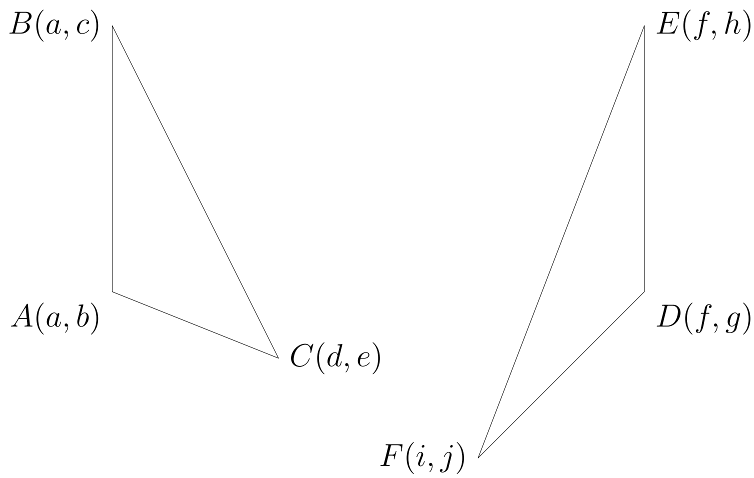

FIGURE 4. Visualization of triangles made by {1,4,4} (left) and {1,3,3} (right).

As can be seen in Figure 4, triangles made by {1,3,3} are mirror images of triangles made by {1,4,4}. By Lemma 2.1, the proof for triangles for {1,4,4} will be applicable to triangles for {1,3,3} as well, without loss of generality.

Theorem 2.2. *It is impossible to form an equilateral triangle by edges in multiset {1,4,4} in the plane with vertices that are lattice points.*

Proof. Assume that the triangle formed by edges in multiset {1,4,4} is an equilateral triangle. In particular, in Figure 5, assume $\triangle AHB$ is an equilateral triangle, which in turn means that all angles are equal to 60°. That is, $\angle AHB = \angle HBA = \angle BAH = 60°$.

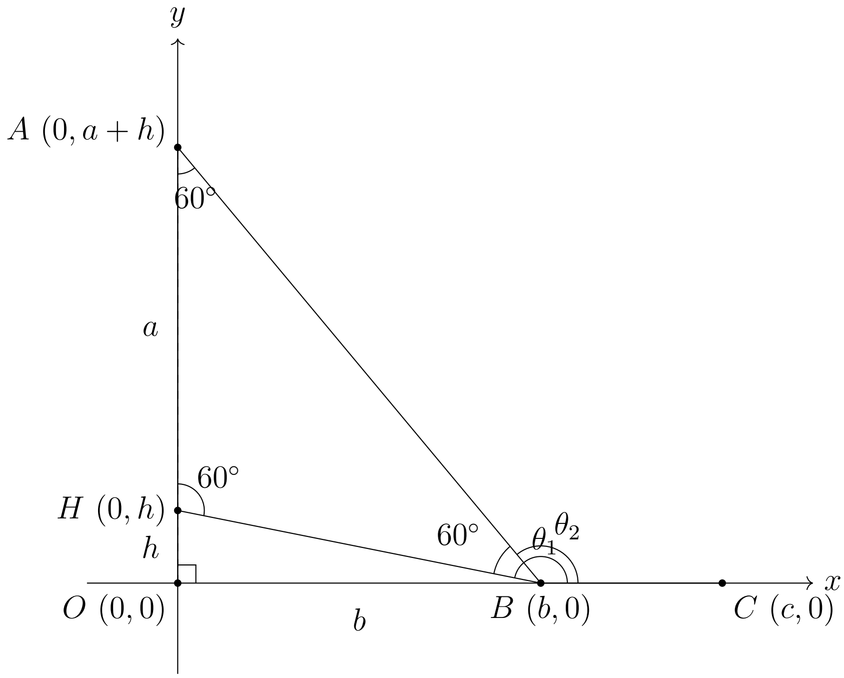

FIGURE 5. Diagram for Theorem 2.2.

$\triangle AHB$ is a general triangle formed by edges in multiset {1,4,4} because $\overline{AH}$ is a vertical line (category 1) and $\overline{BH}$ and $\overline{BA}$ are lines with negative slopes (category 4). In this plane, $a, b, h \in \mathbb{Z} \setminus \{0\}$, since they are lattice points. They cannot equal to zero since they all have assigned lengths as can be seen in Figure 5.[3] Let $\angle CBA = \theta_2$ and $\angle CBH = \theta_1$. Both θ_1 and θ_2 lie in the same quadrants given that they have negative slopes. Then, it should follow that $\theta_1 - \theta_2 = 60°$, in which case, $\tan(\theta_1 - \theta_2) = \tan(60°) = \sqrt{3}$. The slope of $\overline{BH} = \frac{0-h}{b-0} = -\frac{h}{b}$. Then, from the definition of tangent, it can be inferred that $\tan(\theta_1) = -\frac{h}{b}$. Similarly, since the slope of $\overline{BA} = \frac{0-(a+h)}{b-0} = -\frac{a+h}{b}$, it is the case that $\tan(\theta_2) = -\frac{a+h}{b}$. For angles α, β, the difference angle formula for tangent states that:

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \tag{2.1}$$

It should be noted that (2.1) cannot hold when $1 + \tan\alpha\tan\beta = 0$, or in other words, $\tan\alpha\tan\beta = -1$. This case occurs only when the two edges are perpendicular (when the angle difference is 90°), hence the denominator cannot be zero in any of the cases presented moving forward.

[3]If $a = 0$, then the triangle collapses, as $\overline{AB} = \overline{HB}$. If $b = 0$, the triangle collapses again, as $\overline{AO} = \overline{AB}$. Finally, if $h = 0$, then $\overline{HB}$ becomes horizontal, which would render the edge to be a category 2 edge. In addition, then $\triangle AHB \equiv \triangle AOB$, meaning that the triangle simply becomes a right triangle.

Utilizing the above formula for θ_1 and θ_2 gives us:

$$\begin{aligned}
\tan(\theta_1 - \theta_2) &= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \\
&= \frac{-\frac{h}{b} - \left(-\frac{a+h}{b}\right)}{1 + \left(-\frac{h}{b}\right)\left(-\frac{a+h}{b}\right)} \\
&= \frac{-\frac{h}{b} + \frac{a+h}{b}}{1 + \frac{h(a+h)}{b^2}} \\
&= \frac{\frac{a}{b}}{\frac{b^2+h(a+h)}{b^2}} \\
&= \frac{a}{b} \cdot \frac{b^2}{b^2 + h(a+h)} \\
&= \frac{ab}{b^2 + h(a+h)}
\end{aligned}$$

Since $\tan(\theta_1 - \theta_2) = \sqrt{3}$, the following must hold:

$$\frac{ab}{b^2 + h(a+h)} = \sqrt{3}$$

However, this is a contradiction, given that the numerator is a product of integers, which can only be an integer, and that the denominator is a sum of a square of integer, which can only be an integer, and product of integers, which can only be an integer as well. This means that the left hand side is a rational number, given that the numerator and denominator are both integers, and the right hand side is an irrational number. Therefore, it is impossible to construct an equilateral triangle in the plane with vertices that are lattice points.

□

Without loss of generality, the same kind of proof can be employed for triangles with edges from {2,4,4} and {2,3,3}. As seen in Figure 6, both triangles from {2,4,4} and {2,3,3} can be achieved by some kind of counterclockwise or clockwise 90° rotation from triangles in Figure 5. Therefore, by Lemma 2.1, the same kind of proof strategy should be applicable to these triangles as well.

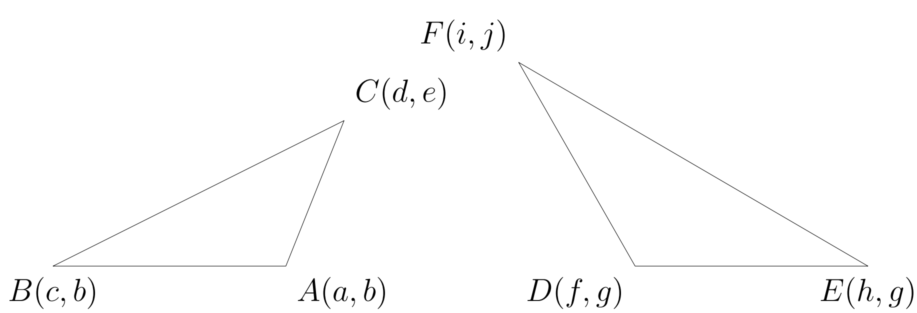

FIGURE 6. Visualization of triangles made by {2,3,3} (left) and {2,4,4} (right).

2.3. **Case 3.** Similarly to Case 2, I will first visualize the two triangles that are grouped together to demonstrate why they are to be included in the same case.

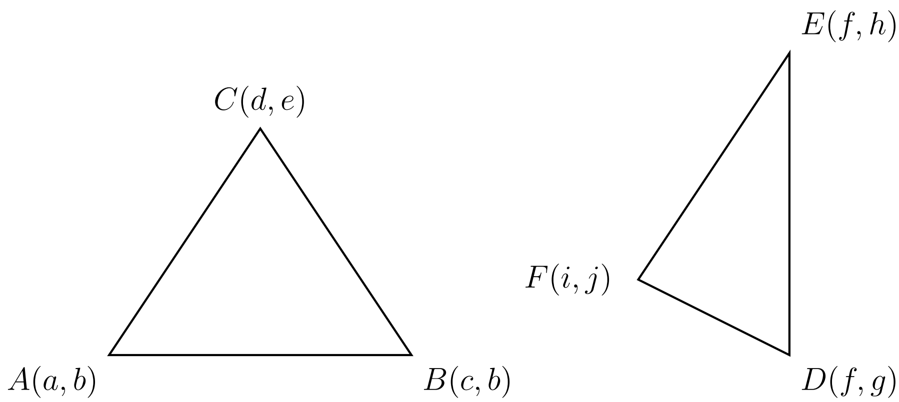

FIGURE 7. Visualization of triangles made by {1,3,4} (left) and {2,3,4} (right).

As can be seen in Figure 7, triangles made by {1,3,4} are $90°$ rotated images of triangles made by {2,3,4}. By Lemma 2.1, the proof for triangles for {1,3,4} will be applicable to triangles for {2,3,4} as well, without loss of generality.

Theorem 2.3. *It is impossible to form an equilateral triangle by edges in multiset {1,3,4} in the plane with vertices that are lattice points.*

Proof. Assume that the triangle formed by edges in multiset {1,3,4} is an equilateral triangle. In particular, in Figure 8, assume $\triangle CHO$ is an equilateral triangle, which in turn means that all angles are equal to $60°$. That is, $\angle CHO = \angle HOC = \angle OCH = 60°$.

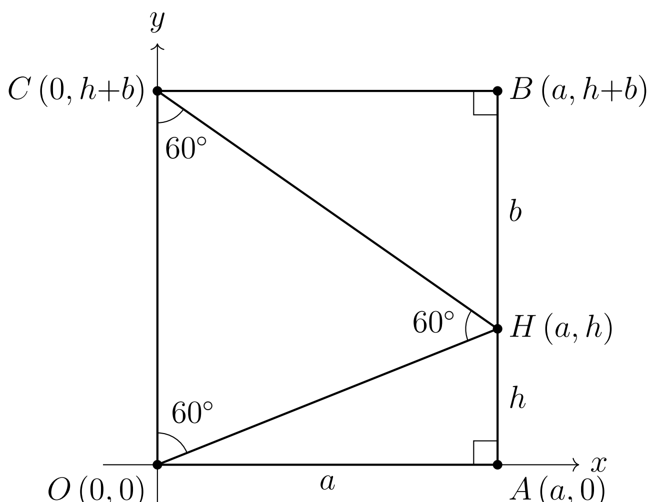

FIGURE 8. Diagram for Theorem 2.3.

$\triangle CHO$ is a general triangle formed by edges in multiset {1,3,4} because $\overline{OC}$ is a vertical line (category 1), $\overline{HO}$ is a line with positive slope (category 3), and $\overline{CH}$ is a line with negative slope (category 4). In this plane, $a, b, h \in \mathbb{Z} \setminus \{0\}$, since they are lattice points. They cannot equal to zero since they all have assigned lengths as can be seen in Figure 8.[4] Given that $\angle HOC = 60°$ and $\angle COA = 90°$, $\angle HOA = \angle COA - \angle HOC = 90° - 60° = 30°$. Then, $\angle AHO = 180° - \angle HOA - \angle HAO = 180° - 30° - 90° = 60°$. Given that $\tan 60° = \sqrt{3}$, $a = \tan \angle AHO \cdot h = \tan 60° \cdot h = \sqrt{3}h$. Since $a = \sqrt{3}h$, the x-coordinate of the vertex H is also $\sqrt{3}h$:

$$a = \sqrt{3}h$$
$$\frac{a}{h} = \sqrt{3}$$

However, it has been established that $a, h \in \mathbb{Z} \setminus \{0\}$, so this is a contradiction: the left hand side is a rational number, since the numerator and the denominator are both integers, and the right hand side is an irrational number. Therefore, it is impossible to construct an equilateral triangle in the plane with vertices that are lattice points.

$\square$

2.4. **Case 4.** Similarly to the previous cases, I will first visualize the two triangles that are grouped together to demonstrate why they are to be included in the same case.

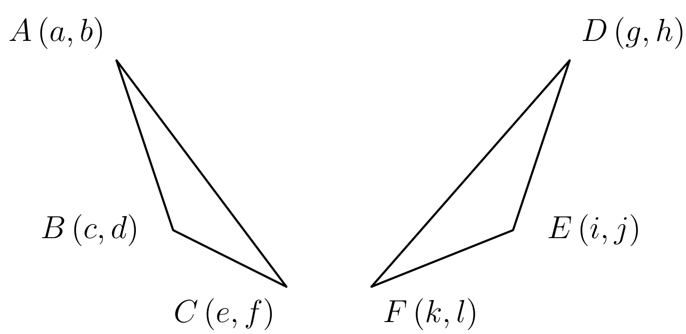

FIGURE 9. Visualization of triangles made by {4,4,4} (left) and {3,3,3} (right).

As can be seen in Figure 9, triangles made by {4,4,4} are mirror images of triangles made by {3,3,3}. By Lemma 2.1, the proof for triangles for {3,3,3} will be applicable to triangles for {4,4,4} as well, without loss of generality.

Theorem 2.4. *It is impossible to form an equilateral triangle by edges in multiset {3,3,3} in the plane with vertices that are lattice points.*

[4]If $a = 0$, then $\square OABC$ collapses and becomes a line. If $b = 0$, then $\angle OCH = 90°$, rendering $\triangle CHO$ to be a right triangle. If $h = 0$, similarly, $\angle HOC = 90°$, rendering $\triangle CHO$ to be a right triangle again.

Proof. Assume that the triangle formed by edges in multiset {3,3,3} is an equilateral triangle. In particular, in Figure 10, assume $\triangle AOB$ is an equilateral triangle, which in turn means that all angles are equal to 60°. That is, $\angle AOB = \angle OBA = \angle BAO = 60°$.

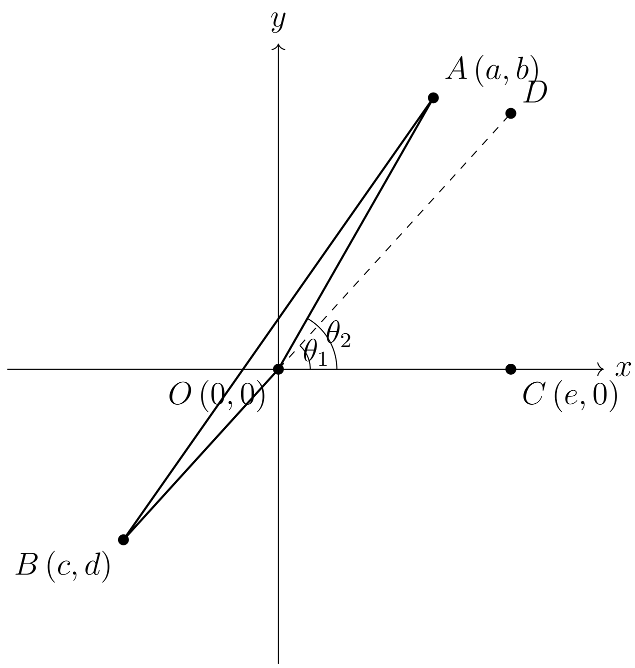

FIGURE 10. Diagram for Theorem 2.4.

$\triangle AOB$ is a general triangle formed by edges in multiset {3,3,3} because $\overline{OA}, \overline{OB}, \overline{AB}$ all have positive slopes (category 3). In this plane, $a, b, c, d, e \in \mathbb{Z} \setminus \{0\}$, since they are lattice points. They cannot equal to zero since they all have assigned coordinate values, as can be seen in Figure 10.[5] Both θ_1 and θ_2 lie in the same quadrants given that they have positive slopes. Given that $\angle AOB = 60°$, $\angle DOA = 180° - \angle AOB = 180° - 60° = 120°$. Let $\angle COD = \theta_1, \angle COA = \theta_2$. Then, $\angle DOA = \theta_2 - \theta_1$, which in turn renders $\theta_2 - \theta_1 = 120°$. Then, $\tan(\theta_2 - \theta_1) = \tan 120° = -\sqrt{3}$. By the definition of theta, $\tan\theta_1 = \frac{d-0}{c-0} = \frac{d}{c}$ and $\tan\theta_2 = \frac{0-b}{0-a} = \frac{b}{a}$. Now, to utilize the difference angle formula from (2.1):

$$\begin{aligned}\tan(\theta_1 - \theta_2) &= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \\ &= \frac{\frac{d}{c} - \frac{b}{a}}{1 + \left(\frac{d}{c}\right)\left(\frac{b}{a}\right)}\end{aligned}$$

[5]If $a = 0$, $\overline{AO}$ becomes a vertical line (category 1). If $b = 0$, $\overline{AO}$ becomes a horizontal line (category 2). If $c = 0$, $\overline{BO}$ becomes a vertical line (category 1). If $d = 0$, $\overline{BO}$ becomes a horizontal line (category 2). If $e = 0$, the vertex 0 becomes equal to C.

$$= \frac{\frac{ad-bc}{ac}}{\frac{ac+bd}{ac}}$$
$$= \frac{ad-bc}{ac+bd}$$

Since $\tan(\theta_1 - \theta_2) = -\sqrt{3}$, the following must hold:

$$\frac{ad-bc}{ac+bd} = -\sqrt{3}$$

However, this is a contradiction, given that the numerator and denominator are both sum or difference of product of integers, which can only be integers. That is, the left hand side is a rational number and the right hand side is an irrational number. Therefore, it is impossible to construct an equilateral triangle in the plane with vertices that are lattice points.

□

2.5. **Case 5.** Similarly to the previous cases, I will first visualize the two triangles that are grouped together to demonstrate why they are to be included in the same case.

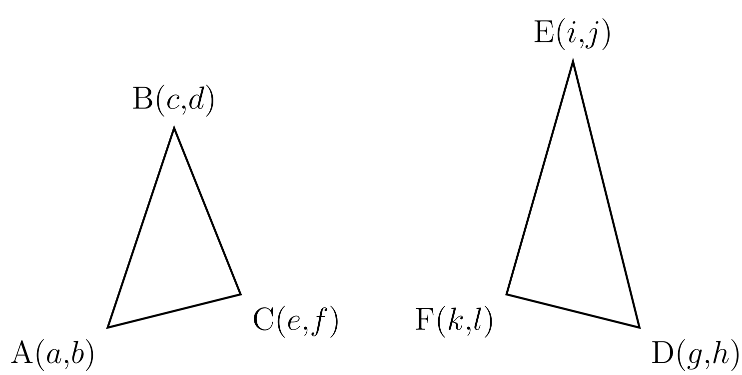

FIGURE 11. Visualization of triangles made by {3,3,4} (left) and {3,4,4} (right).

As can be seen in Figure 11, triangles made by {3,3,4} are mirror images of triangles made by {3,4,4}. By Lemma 2.1, proof for triangles for {3,3,4} will be applicable to triangles for {3,4,4} as well, without loss of generality.

Theorem 2.5. *It is impossible to form an equilateral triangle by edges in multiset {3,3,4} in the plane with vertices that are lattice points.*

Proof. Assume that the triangle formed by edges in multiset {3,3,4} is an equilateral triangle. In particular, in Figure 12, assume $\triangle AOB$ is an equilateral triangle, which in turn means that all angles are equal to 60°. That is, $\angle AOB = \angle OBA = \angle BAO = 60°$.

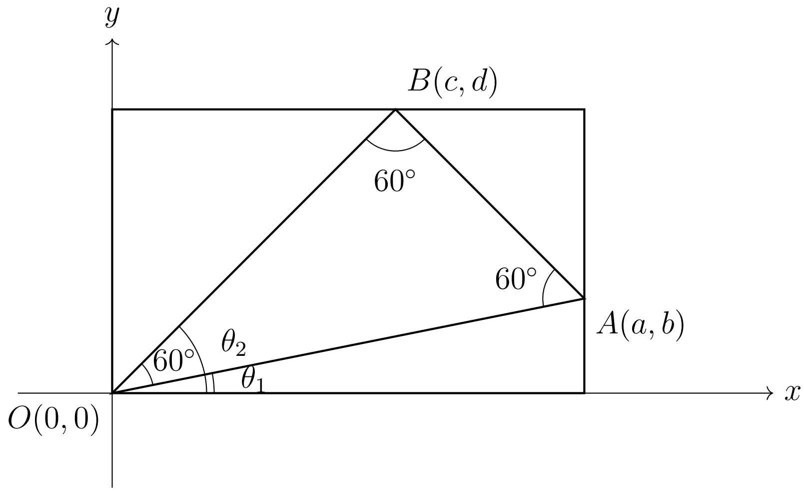

FIGURE 12. Diagram for Theorem 2.5.

$\triangle AOB$ is a general triangle formed by edges in multiset {3,3,4} because $\overline{OA}, \overline{OB}$ both have positive slopes (category 3) and $\overline{BA}$ has a negative slope (category 4). In this plane, $a, b, c, d \in \mathbb{Z} \setminus \{0\}$, since they are lattice points. They cannot equal to zero since they all have assigned coordinate values, as can be seen in Figure 12.[6] Let the angle between $\overline{OA}$ and the x-axis be θ_1 and let the angle between $\overline{OB}$ and the x-axis be θ_2. Both θ_1 and θ_2 lie in the same quadrants given that they have positive slopes. Given that $\angle BOA = 60°$, $\theta_2 - \theta_1 = 60°$. Then, $\tan(\theta_2 - \theta_1) = \tan 60° = \sqrt{3}$. By the definition of theta, $\tan\theta_1 = \frac{b-0}{a-0} = \frac{b}{a}$ and $\tan\theta_2 = \frac{d-0}{c-0} = \frac{d}{c}$. Now, to utilize the difference angle formula from (2.1):

$$\begin{aligned}
\tan(\theta_1 - \theta_2) &= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \\
&= \frac{\frac{b}{a} - \frac{d}{c}}{1 + \left(\frac{b}{a}\right)\left(\frac{d}{c}\right)} \\
&= \frac{\frac{bc-ad}{ac}}{\frac{ac+bd}{ac}} \\
&= \frac{bc - ad}{ac + bd}
\end{aligned}$$

Since $\tan(\theta_2 - \theta_1) = \sqrt{3}$, the following must hold:

$$\frac{bc - ad}{ac + bd} = -\sqrt{3}$$

However, this is a contradiction, given that the numerator and denominator are both sum or difference of product of integers, which can only be integers.

[6]If $a = 0$, $\overline{OA}$ becomes a vertical line (category 1). If $b = 0$, $\overline{OA}$ becomes a horizontal line (category 2). If $c = 0$, $\overline{OB}$ becomes a vertical line (category 1). If $d = 0$, $\overline{OB}$ becomes a horizontal line (category 2).

That is, the left hand side is a rational number and the right hand side is an irrational number. Therefore, it is impossible to construct an equilateral triangle in the plane with vertices that are lattice points.

□

3. Conclusion

Therefore, it is firmly established by Lemma 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.5, and the remark from Case 1, that it is impossible to construct an equilateral triangle in the plane with vertices that are lattice points.

Note that although the proofs for Theorem 2.2, Theorem 2.3, Theorem 2.4, and Theorem 2.5 hinged on specific coordinate choices, the same proofs will follow even if the triangles have arbitrary coordinate choices, as long as they follow the respective categories of edges. For instance, for Theorem 2.5, the vertex O does not necessarily have to be placed on (0,0), but that specific vertex was chosen just for the sake of convenience of computations for the slope values. It could have very well be placed on an arbitrary coordinate, say, (e, f), and the proof would still work.